IEICE TRANS. FUNDAMENTALS, VOL.E104–A, NO.3 MARCH 2021

LETTER

Constructions and Some Search Results of Ternary LRCs with $d = 6$*

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SUMMARY Locally repairable codes (LRCs) are a type of new erasure codes designed for modern distributed storage systems (DSSs). In order to obtain ternary LRCs of distance 6, firstly, we propose constructions with disjoint repair groups and construct several families of LRCs with $1 \leq r \leq 6$, where codes with $3 \leq r \leq 6$ are obtained through a search algorithm. Then, we propose a new method to extend the length of codes without changing the distance. By employing the methods such as expansion and deletion, we obtain more LRCs from a known LRC. The resulting LRCs are optimal or near optimal in terms of the Cadambe-Mazumdar (C-M) bound.

key words: *locally repairable codes, parity-check matrix, disjoint repair groups*

1. Introduction

The coming era of big data brings great pressure to modern distributed storage systems in terms of efficiency and reliability. The traditional triple replication scheme is no longer applicable due to its low efficiency [1], [2]. Therefore, locally repairable codes that can effectively solve such problems have emerged in recent years. When a storage node fails to function temporally or permanently, it can be recovered by accessing at most r other nodes. The number r is called the locality of the node. An LRC with locality r means that all nodes have locality at most r.

Definition 1: An $[n, k, d; r]_q$ LRC is a linear code of length n, dimension k, distance d and locality r over the finite field $\mathbb{F}_q$.

Gopalan et al. has proposed an upper bound of LRCs [3]:

$$d \leq n - k - \lceil \frac{k}{r} \rceil + 2, \tag{1}$$

which is also called the Singleton-like bound since it reduces to the classical Singleton bound when $r = k$. Ternary LRCs reaching the Singleton-like bound has been given in [4]. The Singleton-like bound is not tight for codes over small finite fields. Cadambe and Mazumdar proposed a field-dependent bound as follows [5]:

An $[n, k, d; r]_q$ LRC satisfies:

$$k \leq min_{t\in Z^+}\{tr + k^q_{opt}(n - t(r + 1), d)\}, \tag{2}$$

where Z^+ denotes positive integer and $k^q_{opt}(n, d)$ is the largest possible dimension of a code with length n, distance d, and alphabet size q. The bound (2) is called the Cadambe-Mazumdar (C-M) bound. We use $[n, k, 6; r]$ to denote ternary LRCs with locality r in this paper. An LRC reaching the C-M bound is said to be optimal. If the dimension of LRCs are one less than the C-M bound, we call these LRCs almost optimal.

LRC with small field size is easier to be implemented on hardware, so it has been widely studied by scholars. Gopraju et al. [6] analyzed binary cyclic LRCs. Ernvall et al. proposed LRCs over a small alphabet in [7]. Yang et al. studied the locality of low dimensional ternary optimal codes in [7]–[9]. References [10]–[12] utilized t-spreads to construct binary LRCs of 6 with disjoint repair groups. Hao et al. [13] proposed a class of LRCs with $q \geq r - 1, q = 2^{n-k}$ and $d = 4$ and found four classes of binary LRCs, which reach the Singleton-like bound. Chen et al. [14] proposed improved bounds and constructed some LRCs with distance 5 and 6. Calderbank and Fishburn studied maximal three-independent subsets over $\mathbb{F}^n_3$ in [15].

In this paper, we propose a construction for ternary LRCs of distance 6 with disjoint repair groups inspired by Ref. [11]. Construction conditions for different r are classified and discussed, then we construct several families of LRCs with $1 \leq r \leq 6$. To obtain more LRCs, some methods such as deleting and expanding the parity-check matrices, are also discussed.

An outline of this paper is as follows. In Sect. 2, we introduce some mathematical notations and definitions. Section 3 provides the construction of LRCs with disjoint repair groups, hence, we construct some optimal or almost optimal ternary LRCs. In Sect. 4 we propose a method of expanding the code length n without changing distance and dimension of parity-check matrix. Section 5 concludes the paper.

2. Preliminaries

First we give some mathematical notations and definitions which will be used later:

- Let $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ and v_i be the i-th coordinate of $\mathbf{v}$. The support of $\mathbf{v}$ is denoted as supp($\mathbf{v}$)={i

Manuscript received June 28, 2020.
Manuscript publicized September 1, 2020.

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*This work is supported by National Natural Science Foundation of China (Nos.11801564,11901579) and the Graduate Scientific Research Foundation of Department of Basic Sciences.

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DOI: 10.1587/transfun.2020EAL2070

$| v_i \neq 0\}$ and the Hamming weight of $\mathbf{v}$ is denoted as $\mathrm{wt}(\mathbf{v})=|\mathrm{supp}(\mathbf{v})|$.

- Let I_n be the $n \times n$ identity matrix, and $\mathbb{F}_3 = \{0, 1, 2\}$. $\mathbf{0}_n$, $\mathbf{1}_n$ and $\mathbf{2}_n$ denote the all-zero row vector, all-one row vector and all-two row vector of length n.
- Let $[i] = \{1, 2, \cdots, i\}$ and $[a, b] = \{a, a+1, \cdots, b \mid a \leq b\}$, i, a and b are three positive integers.
- For a linear code with an $(n-k)\times n$ parity-check matrix H, denote the i-th row of H as $\mathbf{h}_i$ ($i \in [n-k]$).
- Let $\otimes$ denote tensor product, and $\mathbb{F}_q^m$ denotes all of m-dimensional vector spaces in finite field $\mathbb{F}_q$.

Definition 2: ([11]) We say an $[n, k, d; r]$ LRC has disjoint repair groups if there exists a set of local check $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_l \in H$ such that $\cup_{i=1}^{l}\mathrm{supp}(\mathbf{h}_i) = [n]$, $wt(\mathbf{h}_i) = r+1$ and $\mathrm{supp}(\mathbf{h}_i) \cap \mathrm{supp}(\mathbf{h}_j) = \emptyset$ for $1 \leq i \neq j \leq l$.

Definition 3: A generator matrix of 2-dimensional simplex code is

$$S_2 = \begin{pmatrix} 1011 \\ 0112 \end{pmatrix},$$

a generator matrix of k-dimensional simplex code can be constructed through recursion formula

$$S_k = \begin{pmatrix} S_{k-1} & \mathbf{0}_{k-1}^{\top} & S_{k-1} & S_{k-1} \\ \mathbf{0}_{n_{k-1}} & 1 & \mathbf{1}_{n_{k-1}} & \mathbf{2}_{n_{k-1}} \end{pmatrix},$$

where n_{k-1} is the number of columns in S_{k-1}. It is easy to know that S_k has $\frac{3^k-1}{2}$ columns.

Definition 4: ([14], [16]) Denote $\mathcal{G}_q(m, s)$ as all s-dimensional subspaces of $\mathbb{F}_q^m$. A subset $\mathcal{S} \subset \mathcal{G}_q(m, s)$ is called a sunflower if any two distinct element in $\mathcal{S}$ intersect in the same t-subspace Z. Moreover, the t-subspace Z is called the center of $\mathcal{S}$ denoted by $Cen(\mathcal{S})$.

3. Construction of LRCs with Disjoint Repair Groups

In this section, we will construct ternary LRCs of distance 6 with disjoint repair groups. Corresponding construct conditions are given for $r \in [2]$. If $r \geq 3$, we first give uniform constraints, then algorithm 1 is proposed to construct LRCs, conveniently.

Let C be an $[n, k, 6; r]$ LRC code. We assume that the parity-check matrix H of C consists of two parts:

$$H = \begin{pmatrix} H_L \\ H_G \end{pmatrix}.$$

As the local check, the upper block H_L is designed to guarantee the locality r of the code C. Suppose that $l = \frac{n}{r+1}$, where n is the code length of C. We generally design H_L as $I_l \otimes \mathbf{1}_{r+1}$. If we regard m as the number of rows of H and let $u = m-l$, then H_G can be represented as an $u\times n$ matrix. The function of H_G is that determines the minimum distance of C. Moreover, the parity-check matrix H can be represented as follows [10]–[12]:

$$H = \begin{pmatrix} \mathbf{1}_{r+1} & \mathbf{0}_{r+1} & \cdots & \mathbf{0}_{r+1} \\ \mathbf{0}_{r+1} & \mathbf{1}_{r+1} & \cdots & \mathbf{0}_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{r+1} & \mathbf{0}_{r+1} & \cdots & \mathbf{1}_{r+1} \\ H_G^1 & H_G^2 & \cdots & H_G^l \end{pmatrix}, \qquad (3)$$
$$\quad H_1 \quad H_2 \quad \cdots \quad H_l$$

where H_G^i ($i \in [l]$) denotes the i-th $u \times (r+1)$ sub-matrix of H_G. Because each column of H_L exists and only exists one non-zero element, it follows the conditions that $\cup_{i=1}^{l}\mathrm{supp}(\mathbf{h}_i) = [n]$, $wt(\mathbf{h}_i) = r+1$ and $\mathrm{supp}(\mathbf{h}_i) \cap \mathrm{supp}(\mathbf{h}_j) = \emptyset$ for $1 \leq i \neq j \leq l$. Hence, we partition H into l disjoint repair groups, which are denoted as $H_1, H_2, \cdots, H_l$.

Given a linear code with a parity-check matrix H, it is well known that its distance is 6 if and only if any 5 columns in H are linearly independent. Since H has been divided into l disjoint repair groups, we can prove $d = 6$ through proving any 5 columns picked from these groups are linearly independent, which will make it easier to classify and discuss. By the elementary row transformations of the matrix, H_G^i can be further transformed into the following format:

$$H_G^i = \begin{pmatrix} \mathbf{0}_u^T & G^i \end{pmatrix},$$

where G^i consists of the r non-zero column vectors in the H_G^i, and called basic vectors set of the H_G^i.

3.1 LRCs with $r \in [2]$

The construction of parity-check matrix as (3) is used here. N_{u_r} as the maximum number of disjoint repair groups for given r and u. Our work includes some previous results. Reference [4] has given the constructions of $[8, 2, 6; 1]$, $[9, 3, 6; 2]$ and $[12, 5, 6; 2]$. In addition, a [39,22,6;2] LRC is proved to exist in [14].

Lemma 1: Let $r = 1$, $u \geq 2$ and $N_{u_1} = \frac{3^u-1}{2}$. For $i \neq j \leq [N_{u_1}]$, $\lambda_1, \lambda_2 \in [2]$, if α^i in G^i and α^j in G^j satisfy $\lambda_1\alpha^i + \lambda_2\alpha^j \neq 0$, then there are $[2l, l-u, 6; 1]$ LRCs for $l \in [u+1, N_{u_1}]$,

Proof We use $(2, 1)$ to denote picking two columns from a disjoint group and one column from the other group. Similarly, when picking any two or three columns in H, it is easy to know that they are linearly independent. When picking four columns in H, there are 3 cases $(1, 1, 1, 1)$, $(2, 1, 1)$ and $(2, 2)$, respectively; when picking five columns in H, there are 3 cases $(1, 1, 1, 1, 1)$, $(2, 1, 1, 1)$ and $(2, 2, 1)$, respectively. Among these cases, (2,2) is linearly independent according to $\lambda_1\alpha^i + \lambda_2\alpha^j \neq 0$, the linear independence of $(2, 2, 1)$ can be derived from case $(2, 2)$, other cases are linearly independent obviously. Due to any 5 columns are linearly independent in H, the distance can be proved to be 6. For the range of l, since the dimensions of LRCs have to be greater than or equal to r, so the minimum value of l is $u+1$. And we find that the vectors of simplex code satisfy the inequality condition, as a result, $N_{u_1} = \frac{3^u-1}{2}$. □

Example 1: If $u = 3$, $l = 5$, there exists an almost optimal $[10, 2, 6; 1]$ LRC, whose parity-check matrices H is

$$H = \begin{pmatrix} 1100000000 \\ 0011000000 \\ 0000110000 \\ 0000001100 \\ 0000000011 \\ 0000010101 \\ 0001000100 \\ 0100000001 \end{pmatrix}.$$

Lemma 2: Let $r = 2$, $u \geq 3$ and $N_{u_2} = \frac{3^{u-1}-1}{2}$. If the following conditions are satisfied, then there are $[3l, 2l-u, 6; 2]$ LRCs for $l \in [\lceil \frac{u+2}{2} \rceil, N_{u_2}]$.

1. The column vectors among each G^i $(i \in [N_{u_2}])$ are linearly independent;

2. For $i \neq j \in [N_{u_2}]$, $\lambda_1, \lambda_2 \in \mathbb{F}_3$, $\lambda_2 \neq 0$, two different columns α_a^i, α_b^i in G^i and two different columns α_x^j, α_y^j in G^j:

(1) $\alpha_a^i + \lambda_1 \alpha_b^i \neq \lambda_2 \alpha_x^j$;

(2) $\alpha_a^i + \lambda_2 \alpha_b^i \neq \alpha_x^j + 2\alpha_y^j$;

Proof The proof of $d = 6$ is similar to the proof in [14] and will not be repeated here. For the minimum value of l, since $2l - u$ has to be greater than 2, so $l > \lceil \frac{u+2}{2} \rceil$. By looking at the inequalities of condition 2, we can see that the space based on α_a^i, α_b^i, and the space based on α_x^j, α_y^j only exist an intersect vector $\alpha_a^i + \alpha_b^i = \alpha_x^j + \alpha_y^j$. If we regard $\alpha_a^i + \alpha_b^i$ as $Cen(\mathcal{S})$, then, $t = 1$. For $\mathcal{G}_q(m, s)$, let $m = u$, $s = 2$, a sunflower $\mathcal{S}$ in $\mathcal{G}_q(u, 2)$ is the set of lines passing through a given point in the finite projective plane $PG(u, q)$. Therefore, the size of the sunflower $\mathcal{S}$ in $\mathcal{G}_q(u, 2)$ is $\frac{q^{u-1}-1}{q-1}$. When $q = 3$, the size is $\frac{3^{u-1}-1}{2}$. Hence, the range of l is $[\lceil \frac{u+2}{2} \rceil, N_{u_2}]$. □

Example 2: If $u = 4$, $l = 4$, there exists an almost optimal $[12, 4, 6; 2]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 111000000000 \\ 000111000000 \\ 000000111000 \\ 000000000111 \\ 001001001010 \\ 001001010001 \\ 001010001001 \\ 010001001001 \end{pmatrix}.$$

3.2 LRCs with $r \geq 3$

In this section, we give uniform conditions for constructing LRC with $r \geq 3$. Following an proposed search algorithm, some kinds of optimal and almost optimal LRCs with $3 \leq r \leq 6$ can be obtained. Here we assume N_{u_r} as obtained maximum number of disjoint repair groups for given r and u by the search algorithm.

Theorem 1: Let $r \geq 3$ and $u \geq 4$. If the following conditions are satisfied, then there are $[l(r+1), lr-u, 6; r]$ LRCs for $l \in [\lceil \frac{u+r}{r} \rceil, N_{u_r}]$.

1. Any k columns in an disjoint repair group are linearly independent;

when $r = 3, k = 4$;

when $r \geq 4, k = 5$;

2. For $i \neq j \in [N_{u_r}]$, $\lambda_1, \lambda_2 \in \mathbb{F}_3$, $\lambda_2 \neq 0$, any three different columns like $\alpha_a^i, \alpha_b^i, \alpha_c^i$ in G^i and any two different columns like α_x^j, α_y^j in G^j:

(1) $\alpha_a^i + \lambda_1 \alpha_b^i \neq \lambda_2 \alpha_x^j$;

(2) $\alpha_a^i + \lambda_2 \alpha_b^i \neq \alpha_x^j + 2\alpha_y^j$;

(3) $\alpha_a^i + \alpha_b^i + \alpha_c^i \neq \lambda_2 \alpha_x^j$;

(4) $\alpha_a^i + \alpha_b^i + \alpha_c^i \neq \alpha_x^j + 2\alpha_y^j$;

Proof When picking any two or three columns in H, it is easy to know that they are linearly independent according to condition 1. When picking four columns in H, there are 5 cases, (4), $(3, 1)$, $(1, 1, 1, 1)$, $(2, 1, 1)$, and $(2, 2)$, respectively; The first four are linearly independent obviously, the fifth case is linearly independent according to $\alpha_a^i + \lambda_1 \alpha_b^i \neq \lambda_2 \alpha_x^j$ and $\alpha_a^i + \lambda_2 \alpha_b^i \neq \alpha_x^j + 2\alpha_y^j$. When picking five columns in H, there are 6 cases, $(1, 1, 1, 1, 1)$, $(2, 1, 1, 1)$, $(2, 2, 1)$, $(3, 1, 1)$, $(4, 1)$ and $(3, 2)$, respectively. The first five possibilities can be derived from the previous proof, and the sixth case is linearly independent according to $\alpha_a^i + \alpha_b^i + \alpha_c^i \neq \lambda_2 \alpha_x^j$ and $\alpha_a^i + \alpha_b^i + \alpha_c^i \neq \alpha_x^j + 2\alpha_y^j$, so we can prove $d = 6$. Since the dimension of LRC needs to be greater than the locality, it is easy to know that the value range of l is $[\lceil \frac{u+r}{r} \rceil, N_{u_r}]$. □

As r and u increase, one can see that it is difficult to determine N_{u_r} and get the r non-zero columns of H_G^i. So, a search algorithm is proposed to solve this problem. Here are some related concepts:

- V_u represents a set, which includes u-dimensional space vectors without zero vector.
- G_m^i: the vectors generated by G^i, mainly includes the following three parts: G^i, $2G^i$, the sum of any one basis vector and 2 times of another basis vector in G^i.
- G_M: all vectors of each G_m^i $(i \in [l])$.
- G_n^i: the vectors generated by G^i, mainly includes the sum of any two or three basis vectors and the vectors that are twice of the sum.
- G_N: all vectors of each G_n^i $(i \in [l])$.
- G_N^*: the set of vectors $V_u \backslash G_M$.
- G_M^*: the set of vectors $V_u \backslash (G_N \cup G_M)$.
- r_u: a set of u-dimensional column vectors that contain r columns.

Corollary 1: For an $[l(r+1), lr-u, 6; r]$ LRC, assume an r_u as G^j $(j = l+1)$, only if the following conditions are met:

1. Any k columns in H^j are linearly independent;

when $r = 3, k = 4$;

when $r \geq 4, k = 5$;

2. G_m^j is disjoint with G_m^i and G_n^i $(i \in [l])$;

3. G_n^j is joint with G_n^i and disjoint with G_m^i $(i \in [l])$;

then the r_u can be regarded as a new basic vectors group, and an $[(l+1)(r+1), r(l+1)-u, 6; r]$ LRC can be constructed

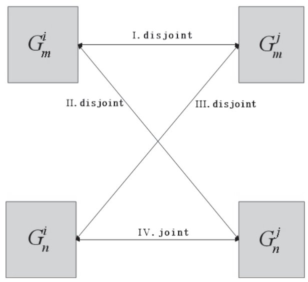

Fig. 1 Joint and disjoint relationship diagram.

by adding G^j to H_G^j.

Proof According to the conditions of Corollary 1, we give a relationship diagram, as shown in Fig. 1. For $\lambda_1 \in \mathbb{F}_3$, $\lambda_2 \in [2]$. Relationship I can derive 3 inequations: (1) $\alpha_a^i \neq \lambda_2\alpha_x^j$; (2) $\alpha_a^i + 2\alpha_b^i \neq \lambda_2\alpha_x^j$; (3) $\alpha_a^i + 2\alpha_b^i \neq \alpha_x^j + 2\alpha_y^j$. Relationships II and III can derive 4 inequations: (1) $\alpha_a^i + \alpha_b^i \neq \lambda_2\alpha_x^j$; (2) $\alpha_a^i+\alpha_b^i \neq \alpha_x^j+2\alpha_y^j$; (3) $\alpha_a^i+\alpha_b^i+\alpha_c^i \neq \lambda_2\alpha_x^j$; (4) $\alpha_a^i+\alpha_b^i+\alpha_c^i \neq \alpha_x^j + 2\alpha_y^j$. Relationship IV is an additional remark. So, the conditions of Corollary 1 are similar to the conditions of Theorem 1. □

According to the definitions of G_M^* and G_N^*, conditions 2 in Corollary 1 imply $G_m^j \subseteq G_M^*$ and conditions 3 in Corollary 1 imply $G_n^j \subseteq G_N^*$. Therefore, we can construct LRCs with $r \geq 3$ using Algorithm 1 as follows.

1. Set r and u ($r \geq 3, u \geq 4$);
2. Let $l = 1, i = 1, G_M^* = V_u, G_N^* = V_u$;
3. While:
4. Choose $r_u \in G_M^*$ as G^l and add to H_G^l;
5. $G_m^l = \{\lambda g_a, \lambda(g_a+2g_b) \mid a \neq b, \lambda \in [2], g_a, g_b \in G^l\}$;
6. $G_n^l = \{\lambda(g_a + g_b), \lambda(g_a + g_b + g_c) \mid a \neq b \neq c, \lambda \in [2], g_a, g_b, g_c \in G^l\}$;
7. If
   (1)Any k columns in H_G^l are linearly independent;
   when $r == 3, k = 4$;
   when $r \geq 4, k = 5$;
   (2)$G_m^l \subseteq G_M^*$;
   (3)$G_n^l \subseteq G_N^*$,
8. Let $i = 1$, update G_M^* and G_N^*;
9. Else
   Let $i = i + 1$
10. If $i > 1000$
11. Break
12. Let $l = l + 1$
13. Let $N_{u_r} = l$
14. Output H_G^i ($i \in [l]$) to construct H.

This is a search algorithm named Algorithm 1. The algorithm mainly transforms the linear restrict conditions of condition 2 in theorem 1 into the judgment of the inclusion relation of the set, thus reducing the computation and the time complexity. The space complexity of the algorithm is mainly determined by the dimension of the vector space searched. Since the judgment conditions of the algorithm is based on the linear relationships between the vectors within 5 columns, when the values of u and r are increased, the algorithm will output more disjoint repair groups, and would terminate when there is no new H_G^l that satisfies the judge conditions after 1000 cycles. Through Algorithm 1, we got some optimal and almost optimal LRCs with $r \in [3, 6]$, see Table 1.

Table 1 Some optimal and almost optimal LRCs for given locality r.

num	r	u	N_{u_r}	$[n, k, 6; r]$
1	3	4	4	$[4l, 3l - 4, 6; 3], 3 \leq l \leq 4$
2	3	5	11	$[4l, 3l - 5, 6; 3], 3 \leq l \leq 11$
3	4	5	4	$[5l, 4l - 5, 6; 4], 3 \leq l \leq 4$
4	4	6	12	$[5l, 4l - 5, 6; 4], 4 \leq l \leq 6$
5	5	4	2	$[6l, 5l - 4, 6; 5], l = 2([4])$
6	5	5	3	$[6l, 5l - 5, 6; 5], 2 \leq l \leq 3$
7	5	6	6	$[6l, 5l - 6, 6; 5], 4 \leq l \leq 6$
8	6	6	4	$[7l, 6l - 6, 6; 6], 3 \leq l \leq 4$

Below we give the specific parity-check matrices H of the codes with the most disjoint repair groups in each class. The parity-check matrices H of other codes in the same class can be obtained by deleting corresponding disjoint repair groups and redundant zero element rows from the H we wrote.

1. When $r = 3, u = 4, l = 4$, there is an $[16, 8, 6; 3]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 1111000000000000 \\ 0000111100000000 \\ 0000000011110000 \\ 0000000000001111 \\ 0120012001210000 \\ 0202021100010020 \\ 0202020200220002 \\ 0211011200210200 \end{pmatrix}.$$

2. When $r = 3, u = 5, l = 11$, there is an $[44, 28, 6; 3]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 11110000000000000000000000000000000000000000 \\ 00001111000000000000000000000000000000000000 \\ 00000000111100000000000000000000000000000000 \\ 00000000000011110000000000000000000000000000 \\ 00000000000000001111000000000000000000000000 \\ 00000000000000000000111100000000000000000000 \\ 00000000000000000000000011110000000000000000 \\ 00000000000000000000000000001111000000000000 \\ 00000000000000000000000000000000111100000000 \\ 00000000000000000000000000000000000011110000 \\ 00000000000000000000000000000000000000001111 \\ 01000001021001210022002102220200001000110112 \\ 00100002000000210202010102200001001102020122 \\ 000100010201002200000201001101110022002100022 \\ 00000100021201210012010002120101010002200211 \\ 00000010001102110222011201000111001200220001 \end{pmatrix}.$$

3. When $r = 4, u = 5, l = 4$, there is an $[20, 11, 6; 4]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 11111000000000000000 \\ 00000111110000000000 \\ 00000000001111100000 \\ 00000000000000011111 \\ 00220021100200201022 \\ 01110000200022101100 \\ 00210010120121202102 \\ 02112000120201002212 \\ 02102011210222102121 \end{pmatrix}.$$

4. When $r = 4, u = 6, l = 6$, there is an $[30, 18, 6; 4]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 111110000000000000000000000000 \\ 000001111100000000000000000000 \\ 000000000011111000000000000000 \\ 000000000000000111110000000000 \\ 000000000000000000001111100000 \\ 000000000000000000000000011111 \\ 022110002200001021010112002200 \\ 002200101000011022110220000011 \\ 021220010202120001110000201201 \\ 022110122102120012000211000002 \\ 011210022200222011200100101201 \\ 000000121101200012120101202121 \end{pmatrix}.$$

5. When $r = 5, u = 4, l = 2$, there is an $[12, 6, 6; 5]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 111111000000 \\ 000000111111 \\ 002121000210 \\ 020220000222 \\ 000201020112 \\ 000111002022 \end{pmatrix}.$$

6. When $r = 5, u = 5, l = 3$, there is an $[18, 10, 6; 5]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 111111000000000000 \\ 000000111111000000 \\ 000000000000111111 \\ 002211021120010021 \\ 001110010002012112 \\ 010110001101021201 \\ 012210000201011002 \\ 001212000102011211 \end{pmatrix}.$$

7. When $r = 5, u = 6, l = 6$, there is an $[36, 24, 6; 5]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 111111000000000000000000000000000000 \\ 000000111111000000000000000000000000 \\ 000000000000111111000000000000000000 \\ 000000000000000000111111000000000000 \\ 000000000000000000000000111111000000 \\ 000000000000000000000000000000111111 \\ 012102022100021201000212002122022211 \\ 002200000002012112021210001201001002 \\ 010120001000020110010101010110020010 \\ 020102010201000011002022011122001120 \\ 021110002122011000002111002002010212 \\ 020010010210022211001120011202020011 \end{pmatrix}.$$

8. When $r = 6, u = 6, l = 4$, there is an $[28, 18, 6; 6]$ LRC, whose a parity-check matrix H is

$$H = \begin{pmatrix} 1111111000000000000000000000 \\ 0000000111111100000000000000 \\ 0000000000000011111110000000 \\ 0000000000000000000001111111 \\ 0100012011212201211010002122 \\ 0222020011122202000220222202 \\ 0101201021221102221200022022 \\ 0020001020120002121220120002 \\ 0222200001000101010210101122 \\ 0212020020201102012100121000 \end{pmatrix}.$$

4. Expanding Code Length of LRCs

For an $[n, k, 6; r]$ LRC with a parity-check matrix H, there is an algorithm to expand the code length n without changing the distance 6 and the dimension of parity-check matrix. The content of the Algorithm 2 is as follows.

1. Let $i = 0, V_{(0)} = V_m, H_{(0)} = H$. //initialization.
2. While :
3. Set $S_i = \{a | a = \Sigma_{j=1}^{4} \lambda_j \mathbf{v}_j, \lambda_j \in [0, 2], \mathbf{v}_j \in H_{(i)}\}$. //pick a set($S_i$) of column vectors that are linear combinations of the column vectors within 4 columns in $H_{(i)}$.
4. Let $V_{(i+1)} = V_{(i)} \setminus S_i$. //get a set of $V_{(i+1)}$, of which any one column and any 4 columns in $H_{(i)}$ are 5 linearly independent.
5. If $V_{(i+1)} \neq \emptyset$
6. Choose $\mathbf{v}$, $\mathbf{v} \in V_{(i+1)}$ //Pick any a vector $\mathbf{v}$ from $V_{(i+1)}$.
7. Let $H_{(i+1)} = [H_{(i)}, \mathbf{v}]$.
8. $i = i + 1$.
9. Else
10. Output $H_{(i)}$.
11. Break.

If H increases i columns, we can get an $[n+i, k+i, 6; r^*]$ LRC with $r^* \in [r + 1, r + i]$.

Example 3: For an optimal [20,11,6;4] LRC, by using the method of expanding code length from V_9, n can be at most up to 24. The codes with $n \in [21,24]$ are optimal. A parity-check matrix of [24,15,6;7] LRC is

$$H = \left(\begin{array}{c:c} 11111000000000000000 & 0111 \\ 00000111110000000000 & 1011 \\ 00000000001111100000 & 1202 \\ 00000000000000011111 & 2120 \\ 00220021100200201022 & 0111 \\ 01110000200022101100 & 1021 \\ 00210010120121202102 & 1021 \\ 02112000120201002212 & 0001 \\ 02102011210222102121 & 2001 \end{array}\right).$$

Using methods such as expansion and deletion, we can construct some optimal and almost optimal LRCs as follows:

An optimal $[13, 6, 6; 3]$ LRC can be constructed by expanding $[12, 5, 6; 2]$ LRC. An optimal [11,4,6;2] LRC is constructed by deleting $[12, 5, 6; 2]$.

When $r = 3$, $u = 4$, $l = 4$, an optimal $[16, 8, 6; 3]$ LRC is obtained in Table 1; an almost optimal $[14, 6, 6; 3]$ LRC and an optimal $[15, 7, 6; 3]$ LRC can be constructed by deleting $[16, 8, 6; 3]$; optimal LRCs for $n \in [17, 28]$ (expect $n = 18, 19$) are constructed by expanding $[16, 8, 6; 3]$.

When $r = 4$, $u = 5$, $l = 4$, we can construct an optimal $[20, 11, 6; 4]$ LRC in Table 1; almost optimal LRCs for $n \in [18, 19]$ are constructed by deleting $[20, 11, 6; 4]$; optimal LRCs for $n \in [21, 24]$ are constructed by expanding $[20, 11, 6; 4]$.

5. Conclusions

In this paper, we proposed a construction for ternary LRCs based on disjoint repair groups with distance 6. All codes obtained from our construction are optimal or almost optimal according to the C-M bound. In the future, we think that the properties of obtained LRCs have the possibility of further optimization, and we want to construct some optimal LRCs with other distances.

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