PAPER *Special Section on Discrete Mathematics and Its Applications*

Ramsey Numbers of Trails

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SUMMARY We initiate the study of Ramsey numbers of trails. Let $k \geq 2$ be a positive integer. The Ramsey number of trails with k vertices is defined as the the smallest number n such that for every graph H with n vertices, H or the complete $\overline{H}$ contains a trail with k vertices. We prove that the Ramsey number of trails with k vertices is at most k and at least $2\sqrt{k} + \Theta(1)$. This improves the trivial upper bound of $\lfloor 3k/2 \rfloor - 1$.

key words: *Ramsey theory, trails, Eulerian graphs, semi-Eulerian graphs*

1. Introduction

Ramsey theory is one of the topics in discrete mathematics that has been studied over the years [1], [4]. For graphs, the Ramsey number was first studied for complete graphs, and later it was studied for other classes of graphs such as paths, cycles, and trees; a good source of references is given by Radziszowski [5]. Applications of Ramsey theory ranges from pure mathematics such as number theory and harmonic analysis to computer science such as approximation algorithms and complexity theory [3], [6].

For graphs G and G', the Ramsey number of the pair (G, G') is the smallest number n such that for every graph H with n vertices, H contains a copy of G or the complement $\overline{H}$ contains a copy of G'. It is known that for every pair (G, G') of finite graphs, the Ramsey number of (G, G') exists, and the determination of the Ramsey number is the ultimate goal. However, even for complete graphs, the exact Ramsey number is not known: When $G = G' = K_5$ we only know that the Ramsey number lies between 43 and 48 [5].

In this paper, we initiate the study of Ramsey numbers for trails. Unlike paths, trails may have a repetition of vertices. To study the Ramsey number of trails, we first fix the number of vertices in a trail. Let k and ℓ be integers. Then, the Ramsey number of trails with k vertices and ℓ vertices is defined as the smallest number n such that for every graph H with n vertices, H contains a trail with k vertices or $\overline{H}$ contains a trail with ℓ vertices.

The ultimate goal is to determine the Ramsey number of trails. Unfortunately, we are unable to provide a definite answer. Nonetheless, we give a progress toward the ultimate goal. We concentrate on the diagonal case, i.e., the case where $k = \ell$. Our main theorems give an improved upper bound of k, and also a lower bound of roughly $2\sqrt{k}$. We note here that a trivial upper bound is $\lfloor 3k/2 \rfloor - 1$, which will be sketched in the next section.

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2. Preliminaries

In this paper, all graphs are finite, simple and undirected. A *graph* G is defined as a pair (V, E) of a finite set V and $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$, where V is the set of *vertices* of G and E is the set of *edges* of G. The *degree* of a vertex $v \in V$ is the number of edges incident to v, i.e., $|\{e \in E \mid v \in e\}|$.

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and $u, v \in V'$ for every $e = \{u, v\} \in E'$. For a graph $G = (V, E)$, the *complement* of G, denoted by $\overline{G}$, is a graph with vertex set V and edge set $\overline{E} = \{\{u, v\} \mid u, v \in V, u \neq v, \{u, v\} \notin E\}$. Namely, $\overline{G} = (V, \overline{E})$. A pair (G, H) of graphs is called *complementary* if $H = \overline{G}$.

A graph is *complete* if each pair of vertices is joined by an edge. The complete graph with n vertices is denote by K_n. A graph $P = (V, E)$ is a *path* if $V = \{v_1, v_2, \ldots, v_n\}$, and $E = \{\{v_i, v_{i+1}\} \mid i \in \{1, 2, \ldots, n-1\}\}$. The path with n vertices is denote by P_n.

A *walk* is a sequence $v_1 e_1 v_2 \ldots e_{k-1} v_k$ of vertices v_i and edges e_i such that for $1 \leq i \leq k$, the edge $e_i = \{v_i, v_{i+1}\}$. Here, k is the number of vertices of the walk, v_1 and v_k are called *endpoints* of the walk. A *trail* is a walk in which all the edges are different from each other. A trail that satisfies $v_1 = v_k$ is called a *circuit*. A graph is *connected* if it has a trail from any vertex to any other vertex.

Let G be a connected graph. An *Eulerian circuit* of G is a circuit of G that passes every edge exactly once. If G has an Eulerian circuit, then G is called *Eulerian*. An *Eulerian trail* of G is a trail of G that passes every edge exactly once. If G has an Eulerian trail but no Eulerian circuit, then G is called a *semi-Eulerian*. It is well-known and easy to prove that a connected graph G is Eulerian if and only if the degree of every vertex of G is even, and G is semi-Eulerian if and only if the number of odd-degree vertices is two.

For $k \geq 1$, we denote by $\mathcal{T}_k$ the set of connected graphs that have an Eulerian circuit or an Eulerian trail with k vertices (see Fig. 1). Note that in our definitions, vertices in trails and circuits are counted multiple times if they are passed multiple times. Therefore, some graphs in $\mathcal{T}_k$ may have less than k vertices.

Let C and C' be two graph classes, i.e., possibly infinite

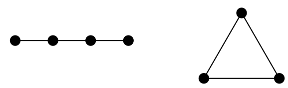

Fig. 1 Graphs in $\mathcal{T}_4$. Note that the right graph has only three vertices, but it has a trail with four vertices.

Table 1 The values of $R(\mathcal{T}_k, \mathcal{T}_k)$.

k	2	3	4	5	6	7	8	9	10
$R(\mathcal{T}_k, \mathcal{T}_k)$	2	3	4	5	6	6	6	7	7

sets of graphs. Then, the *Ramsey number* of C and C' is the smallest number n such that for every graph H with n vertices, H contains a graph in C or $\overline{H}$ contains a graph in C'. The Ramsey number of C and C' is denoted by $R(C, C')$. If C and C' are singletons (i.e., contain only one graph as $C = \{G\}$ and $C' = \{G'\}$), then the Ramsey number of C and C' is denoted by $R(G, G')$.

Gerencsér and Gyárfás [2] determined the exact value of the Ramsey number of paths, as in the following theorem.

Lemma 1 ([2]): Let $k \geq \ell \geq 2$. Then, $R(P_k, P_\ell) = k + \lfloor \ell/2 \rfloor - 1$.

Since P_k belongs to $\mathcal{T}_k$, $R(\mathcal{T}_k, \mathcal{T}_k) \leq R(P_k, P_k) = k + \lfloor k/2 \rfloor - 1 = \lfloor 3k/2 \rfloor - 1$. This is the trivial upper bound mentioned in the previous section.

To gain the first impression, we have conducted a computer search of the Ramsey number $R(\mathcal{T}_k, \mathcal{T}_k)$ for small values of k. This has been performed with the following procedure. For $2 \leq n \leq 7$, we generate all graphs G with n vertices. For each such G, we calculate $\mathrm{t}(G)$, which is defined as the number of vertices in the longest trail in G or $\overline{G}$. Then, we determine value(n), which is defined as the minimum value of $\mathrm{t}(G)$ for all G with n vertices. If k satisfies value($n-1$) $< k \leq$ value(n), then we know that $R(\mathcal{T}_k, \mathcal{T}_k)$ is equal to n.

The result of the computer search is summarized in Table 1. We may observe that the upper bound of $\lfloor 3k/2 \rfloor - 1$ should be improved.

3. Main Theorem: Lower Bound

We begin with a lower bound of $R(\mathcal{T}_k, \mathcal{T}_k)$.

Theorem 1: Let k be a positive integer. Then,

$$R(\mathcal{T}_k, \mathcal{T}_k) \geq \begin{cases} k & \text{if } k \leq 6, \\ \left\lceil \dfrac{1 + \sqrt{16k - 7}}{2} \right\rceil & \text{if } k \geq 7. \end{cases}$$

The rest of the section is devoted to the proof of Theorem 1. We first consider the case when $k \leq 6$.

Let $k = 2$. Then, there is no graph of $\mathcal{T}_2$ in the complete graph K_1. Therefore, $R(\mathcal{T}_2, \mathcal{T}_2) \geq 2$.

Let $k = 3$. Then, the complete graph K_2 has only one edge. So, there is no graph of $\mathcal{T}_3$ in K_2. Thus, $R(\mathcal{T}_3, \mathcal{T}_3) \geq 3$.

Let $k = 4$. Consider the complementary pair of graphs with three vertices as shown in Fig. 2. Since those two graphs have at most two edges, no element of $\mathcal{T}_4$ is contained in either graph. Therefore, $R(\mathcal{T}_4, \mathcal{T}_4) \geq 4$.

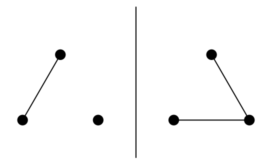

Fig. 2 A complementary pair of graphs that contain no elements of $\mathcal{T}_4$.

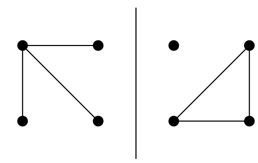

Fig. 3 A complementary pair of graphs that contain no elements of $\mathcal{T}_5$.

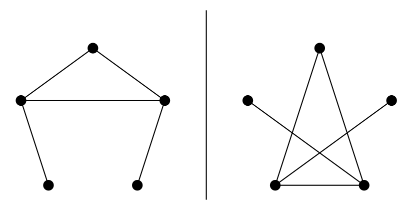

Fig. 4 A complementary pair of graphs that contain no elements of $\mathcal{T}_6$.

Let $k = 5$. Consider the complementary pair of graphs with four vertices as shown in Fig. 3. Since those two graphs have three edges, no element of $\mathcal{T}_5$ is contained in either graph. Hence, $R(\mathcal{T}_5, \mathcal{T}_5) \geq 5$.

Let $k = 6$. Consider the complementary pair of graphs with five vertices as shown in Fig. 4. Those graphs have five edges. Hence, for an element of $\mathcal{T}_6$ to be contained in either of the two graphs, one of the two graphs must be Eulerian or semi-Eulerian. However, each graph has four odd-degree vertices. Thus, these graphs are neither Eulerian nor semi-Eulerian, and have no elements of $\mathcal{T}_6$.

Next, we consider the case where $k \geq 7$. To complete the proof, we use the following two lemmas.

Lemma 2: The number of vertices of a complete graph with $m \geq 0$ edges is $(1 + \sqrt{1 + 8m})/2$.

Proof. Let $n \geq 1$ be the number of vertices of a complete graph with m edges. In this case, $m = n(n-1)/2$. Solving for $n \geq 1$, we have $n = (1 + \sqrt{1 + 8m})/2$. □

Lemma 3: Let $k \geq 7$ and let n be the number of vertices of a complete graph with at most $2k - 2$ edges. Then, there exists a subgraph $G = (V, E_1)$ of K_n such that G and $\overline{G}$ have no element of $\mathcal{T}_k$.

Before proving Lemma 3, we finish the proof of Theorem 1 using Lemmas 2 and 3.

Proof of Theorem 1 when $k \geq 7$. Let $k \geq 7$ and let n be the number of vertices of a complete graph with at most $2k - 1$ edges. Then, $R(\mathcal{T}_k, \mathcal{T}_k) \geq n$ from Lemma 3. Therefore, by

Lemma 2

$$R(\mathcal{T}_k,\mathcal{T}_k) \geq \left\lceil \frac{1+\sqrt{1+8(2k-1)}}{2} \right\rceil = \left\lceil \frac{1+\sqrt{16k-7}}{2} \right\rceil.$$

□

Thus, it suffices to prove Lemma 3.

Proof of Lemma 3. We distinguish the cases $|E| \leq 2k-4$, $|E| = 2k-3$ and $|E| = 2k-2$.

▹ Case 1: $|E| \leq 2k-4$.

Choose $G = (V, E_1)$ as any subgraph with $|E_1| = \lceil |E|/2 \rceil$. Then, since $|E_1| = \lceil |E|/2 \rceil \leq k-2, |E - E_1| = \lfloor |E|/2 \rfloor \leq k-2$, it follows that G and $\overline{G}$ have no element of $\mathcal{T}_k$.

▹ Case 2: $|E| = 2k-3$.

Consider a complete graph K_n with $2k-3$ edges. Then,

$$n = \frac{1+\sqrt{1+8(2k-3)}}{2} = \frac{1+\sqrt{16k-23}}{2} > 5$$

from Lemma 2. Let $v_1, \ldots, v_n$ be the vertices of K_n, and let $E_c = \{\{v_i, v_{i+1}\} \mid i = 1, 2, \ldots, n-1\} \cup \{v_n, v_1\}$. Then, the graph $C = (V, E_c)$ is a cycle contained in K_n. We construct an Eulerian or a semi-Eulerian graph S with $k+1$ edges that contains C.

Before constructing such S, we observe that this is enough for our purpose. Since S contains C and $n \geq 6$, there exist two edges e_1, e_2 of C such that $S' = S - \{e_1, e_2\}$ has exactly four odd-degree vertices. Thus, S' is neither Eulerian nor semi-Eulerian. Since S' has only $k-1$ edges, S' include no element of $\mathcal{T}_k$. Further, $\overline{S'}$ has only $k-2$ edges, and $\overline{S'}$ includes no element of $\mathcal{T}_k$, either.

To find a subgraph S with the desired properties, we further distinguish two cases according to the parity of n.

▹ Case 2-1: n is odd.

Let $G = (V, E - E_c)$. Then, G is Eulerian since the degree of each vertex of G is even and G is connected. Thus, G contains a trail with $|E| - n + 1 = 2k - 2 - n \geq k + 2 - n$ vertices. Let T be a subgraph of G obtained by the first $k + 1 - n$ edges of such a trail, and let $S = C \cup T$. Then, S is Eulerian or semi-Eulerian with $k+1$ edges.

▹ Case 2-2: n is even.

Let $G = (V, E - E_c - \{\{v_i, v_{n/2+i}\} \mid i = 1, 2, \ldots, n/2\})$. Then, G is Eulerian since the degree of each vertex of G is even and G is connected.

Thus, G contains a trail with $|E| - 3n/2 + 1 = 2k - 2 - 3n/2 \geq k + 2 - n$ vertices since $n > 5$ and $n(n-1)/2 = 2k-3$. Let T be a subgraph of G obtained by the first $k+1-n$ edges of such a trail, and let $S = C \cup T$. Then, S is Eulerian or semi-Eulerian with $k+1$ edges.

▹ Case 3: $|E| = 2k-2$.

This case is analogous to Case 2 where $|E| = 2k-3$. Note that for a complete graph K_n with $2k-2$ edges. we have $n = (1 + \sqrt{1+8(2k-2)})/2 = (1 + \sqrt{16k-15})/2 > 5$ from Lemma 2.

We have to take care of the argument after constructing S because $\overline{S'}$ has $k-1$ edges and we need a different argument to show that $\overline{S'}$ includes no element of $\mathcal{T}_k$. Remind that S' contains $k-1$ edges from a trail of G and the edges of C.

We distinguish two cases according to the parity of n. First, let n be odd. Then, the degree of every vertex of K_n is even. Since S' has four odd-degree vertices, $\overline{S'}$ has four odd-degree vertices, too. Thus, $\overline{S'}$ is neither Eulerian nor semi-Eulerian. Since $\overline{S'}$ has only $k-1$ edges, $\overline{S'}$ includes no element of $\mathcal{T}_k$.

Next, let n be even. We first observe that $n \geq 8$. We already know that $n \geq 6$, but if $n = 6$, then the number of edges of K_n is 15, which is not of the form $2k-2$: this is impossible. Therefore, $\overline{S'}$ has at least four odd-degree vertices since S' has $n-4$ even-degree vertices and $n-4 \geq 4$. Thus, $\overline{S'}$ is neither Eulerian nor semi-Eulerian. Since $\overline{S'}$ has only $k-1$ edges, $\overline{S'}$ includes no element of $\mathcal{T}_k$. □

4. Main Theorem: Upper Bound

We already observed that $R(\mathcal{T}_k,\mathcal{T}_k) \leq \lfloor 3k/2 \rfloor - 1$ as a trivial upper bound. Now, we improve the upper bound in the next theorem.

Theorem 2: For every integer $k \geq 2$, it holds that $R(\mathcal{T}_k,\mathcal{T}_k) \leq k$.

To this end, for any graph G with k vertices, we prove either G or its complement $\overline{G}$ contains a trail with k vertices.

We begin with the following lemma which will be used in the proof of the theorem.

Lemma 4: Let $G = (V, E)$ be a bipartite graph with partite sets A and B, i.e., $A \cup B = V$, $A \cap B = \emptyset$ and each edge of G joins a vertex of A and a vertex of B. If $|A| = 3$ and the degree of every vertex of B is two, then G contains a trail such that both endpoints belong to A and the number of edges is $2|B|$.

Proof. Denote the three elements of A by a_1,a_2, and a_3. We distinguish the following two cases according to the existence of an isolated vertex (i.e., a vertex of degree zero) in A.

▹ Case 1: A has an isolated vertex.

Without loss of generality, assume that a_3 is an isolated vertex. Since each vertex in B has degree two, it is adjacent to a_1 and a_2. Hence, the bipartite graph $G' = G - a_3$ is connected. Furthermore, the number of odd-degree vertices in G' is zero or two since the degree of a_1 and a_2 is $|B|$, and the degree of every vertex in B is two. Thus, G' is Eulerian or semi-Eulerian and has $2|B|$ edges. When G' is Eulerian, G' contains a trail with $2|B|$ edges such that both endpoints coincide with a_1. When G' is semi-Eulerian, G' contains a trail with $2|B|$ edges such that one endpoint is a_1 and the other endpoint is a_2.

▹ Case 2: A has no isolated vertex.

Without loss of generality, assume that there exists a vertex $b \in B$ adjacent to a_1 and a_2. Since there is no isolated vertex, there is a vertex $b' \in B$ adjacent to a_3. As the degree of b' is two, b' is adjacent to either a_1 or a_2. Therefore, the three vertices of a_1, a_2 and a_3 are connected by paths. This implies that G is connected since every vertex in B is adjacent to one of a_1, a_2 and a_3. Since G is bipartite and the degree of every vertex in B is two, the sum of the degrees of a_1, a_2 and a_3 is even. If there are an odd number of odd-degree vertices in A, then the sum of the degrees of a_1, a_2 and a_3 is odd, contradicting the fact that the sum of the degrees of a_1, a_2 and a_3 is even. Therefore, the number of odd-degree vertices is zero or two.

When there is no odd-degree vertex, then G is Eulerian, and contains a trail with $2|B|$ edges such that both endpoints coincide with a_1. When there are two odd-degree vertices, let them be a_s and a_t. Then, G is semi-Eulerian, and contains a trail with $2|B|$ edges such that one endpoint is a_s and the other endpoint is a_t. □

We are now ready for the proof of Theorem 2.

Proof of Theorem 2. The proof uses the induction on k. When $k \leq 10$, $R(\mathcal{T}_k, \mathcal{T}_k) \leq k$ holds from Table 1.

Now, fix an arbitrary integer $k \geq 11$ and suppose that the statement is true for $k' < k$. Consider a graph $G = (V, E)$ with k vertices. For a subgraph G' with $k-1$ vertices of G, by induction hypothesis, either G' or $\overline{G'}$ contains a trail S with $k-1$ vertices. If G' contains S, then G contains S because G' is a subgraph of G. If $\overline{G'}$ contains S, then $\overline{G}$ contains S because $\overline{G'}$ is a subgraph of $\overline{G}$. Therefore, either G or $\overline{G}$ contains S. Without loss of generality, suppose G contains S. Let $S = u_1 e_1 u_2 e_2 \ldots e_{k-2} u_{k-1}$ where $e_i = \{u_i, u_{i+1}\}$ for all $i \in \{1, 2, \ldots, k-2\}$, $U = \{u_1, u_2, \ldots, u_{k-1}\}$ be the set of vertices in S, and $W = V - U$. Note that the size of U can be smaller than $k-1$ since some vertices can be identical.

If there exists a vertex $w \in W$ such that $\{u_1, w\} \in E$, then G contains the trail $w\{w, u_1\}S$ with k vertices. Similarly, if there exists a vertex $w \in W$ such that $\{u_{k-1}, w\} \in E$, then G contains the trail $S\{u_{k-1}, w\}w$ with k vertices. If there is a vertex $u \in U$ such that $\{u, u_1\} \in E$ is not included in S, then G contains the trail $u\{u, u_1\}S$ with k vertices. Similarly, if there is a vertex $u \in U$ such that $\{u, u_{k-1}\} \in E$ is not included in S, then G contains the trail $S\{u_{k-1}, u\}u$ with k vertices. In all of these cases, G contains a trail with k vertices and we are done.

Hence, we only need to consider the cases where the following two conditions are satisfied.

Condition 1. For every $w \in W$, $\{u_1, w\} \notin E$ and $\{u_{k-1}, w\} \notin E$. That is, $\{u_1, w\} \in \overline{E}$ and $\{u_{k-1}, w\} \in \overline{E}$.

Condition 2. For every $u \in U$, if $\{u, u_1\}$ is not included S, then $\{u, u_1\} \notin E$. That is, $\{u, u_1\} \in \overline{E}$. If $\{u, u_{k-1}\}$ is not included S, then $\{u, u_{k-1}\} \notin E$. That is, $\{u, u_{k-1}\} \in \overline{E}$.

We distinguish the cases according to the "shape" of S.

▹ Case 1: S is a path.

Since S is a path, S contains no repeated vertex. Therefore, $|U| = k-1$ and $|W| = 1$. Let w be the only vertex in W. Since S contains no repeated vertex, for $3 \leq i \leq k-1$, the edges $\{u_i, u_1\}$ are not included in S. Also, for $1 \leq i \leq k-3$, the edges $\{u_i, u_{k-1}\}$ are not included in S. From Condition 2, $\{u_1, u_{k-1}\} \in \overline{E}$ and for $3 \leq i \leq k-3$, $\{u_1, u_i\} \in \overline{E}$ and $\{u_{k-1}, u_i\} \in \overline{E}$. Consider a subgraph $G' = (V', E')$ of $\overline{G}$, where $V' = V - \{u_2, u_{k-2}\}$ and $E' = \{\{u_1, u_i\}, \{u_{k-1}, u_i\} \mid i \in \{3, 4, \ldots, k-3\}\} \cup \{\{u_1, w\}, \{u_{k-1}, w\}, \{u_1, u_{k-1}\}\}$. Each vertex of V' except u_1 is adjacent to u_1. Hence, G' is connected. Further, since the degree of each vertex in V' except u_1 and u_{k-1} is two, and the degrees of u_1 and u_{k-1} are $|V| - 3$, the number of odd-degree vertices in G' is zero or two. Therefore, G' is Eulerian or semi-Eulerian. Since G' has $2k-7$ edges, G' contains a trail T with $2k-6 \geq k$ vertices. Since G' is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains T.

▹ Case 2: S is a circuit.

When S is a circuit, $u_1 = u_{k-1}$. Therefore, $|U| \leq k-2$ and $|W| = k - |U| \geq 2$. Denote the elements of W by $w_1, w_2, \ldots, w_{|W|}$.

If there exist $w \in W$ and $u_i \in U$ such that $\{w, u_i\} \in E$, then we have a trail

$$T = w\{w, u_i\}u_i e_i u_{i+1} \ldots u_{k-1} e_1 u_2 \ldots u_i$$

since $u_1 = u_{k-1}$. Note that T has k vertices. Therefore, G contains a trail with k vertices.

Hence, we only need to consider the situation where $\{w, u\} \notin E$, i.e., $\{w, u\} \in \overline{E}$ for every $w \in W$ and every $u \in U$. We distinguish two cases according to the comparison of $|U|$ and $|W|$.

▹ Case 2-1: $|U| \geq |W|$.

Choose two vertices $w_1, w_2 \in W$ arbitrarily, and let $V' = U \cup \{w_1, w_2\}$ and $E' = \{\{w_1, u\}, \{w_2, u\} \mid u \in U\} \subseteq \overline{E}$. Consider the subgraph $G' = (V', E')$ of $\overline{G}$. Then, G' is connected since every vertex in U is adjacent to w_1 and w_2. The degree of every vertex in U is two, and the degree of w_1 and w_2 are both $|U|$. Hence, the number of odd-degree vertices in G' is zero or two. Therefore, G' is Eulerian or semi-Eulerian. Since G' has $2|U|$ edges, G' contains a trail T with $2|U| + 1 \geq |U| + |W| + 1 = |U| + (k - |U|) + 1 = k + 1$ vertices. Since G' is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains T.

▹ Case 2-2: $|U| \leq |W|$.

If $|U| < 2$, then the number of vertices in S is less than 1, which contradicts the fact that the number of vertices in S is $k-1 \geq 10$. Therefore, $|U| \geq 2$. Choose two vertices $a, b \in U$ arbitrarily, and let $V' = W \cup \{a, b\}$ and $E' = \{\{w, a\}, \{w, b\} \mid w \in W\} \subseteq \overline{E}$. Consider the subgraph $G' = (V', E')$ of $\overline{G}$. Since every vertex in W is adjacent to a and b, G' is connected. The degree of every vertex in W is two, and the degree of a and b are both $|W|$. Hence, the number of odd-degree vertices in G' is zero or two. Therefore, G' is Eulerian or semi-Eulerian. Since G' has $2|W|$ edges, G' contains a trail T with $2|W|+1 \geq |U|+|W|+1 = |U|+(k-|U|)+1 = k+1$ vertices. Since G' is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains T.

▹ Case 3: S is not a path or a circuit.

Since S is not a path, $|U| \leq k-2$. Since S is not a circuit, $u_1 \neq u_{k-1}$. We distinguish cases according to the size of U.

▹ Case 3-1: $|U| = k-2$.

Since S is a trail with $k-1$ vertices and $|U| = k-2$, there is only one vertex x that is used more than once in S. If $x \neq u_1$ and $x \neq u_{k-1}$, then there are at most two vertices adjacent to either u_1 or u_{k-1} in G. If x is u_1 or u_{k-1}, then there are at most four vertices adjacent to either u_1 or u_{k-1} in G.

Let U' be the set of elements of $U - \{u_1, u_{k-1}\}$ that are not adjacent to either u_1 or u_{k-1} in G. Every vertex $u' \in U'$ satisfies $\{u', u_1\} \notin E$ and $\{u', u_{k-1}\} \notin E$, i.e. $\{u', u_1\} \in \overline{E}$ and $\{u', u_{k-1}\} \in \overline{E}$. Further, $|U'| \geq |U - \{u_1, u_{k-1}\}| - 4 = |U| - 2 - 4 = k - 8$. From Condition 1, for each vertex $w \in W$, we have $\{u_1, w\} \in \overline{E}$ and $\{u_{k-1}, w\} \in \overline{E}$. Let $V' = U' \cup W \cup \{u_1, u_{k-1}\}$, $E' = \{\{w, u_1\}, \{w, u_{k-1}\} \mid w \in W\} \cup \{\{u_1, u'\}, \{u_{k-1}, u'\} \mid u' \in U'\} \subseteq \overline{E}$ and consider the subgraph $G' = (V', E')$ of $\overline{G}$. Since every vertex in $U' \cup W$ is adjacent to u_1 and u_{k-1}, G' is connected. The degree of every vertex in $U' \cup W$ is two, and the degree of u_1 and u_{k-1} are both $|U'| + |W|$. Hence, the number of odd-degree vertices in G' is zero or two. Therefore, G' is Eulerian or semi-Eulerian. Since G' has $2(|U'|+|W|)$ edges, G' contains a trail T with $2(|U'|+|W|)+1 \geq 2(k-8+2)+1 = 2k-11 \geq k$ vertices. Since G' is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains T.

▹ Case 3-2: $|U| \leq \lfloor k/2 \rfloor$.

From Condition 1, for each vertex $w \in W$, we have $\{u_1, w\} \in \overline{E}$ and $\{u_{k-1}, w\} \in \overline{E}$. Let $V' = W \cup \{u_1, u_{k-1}\}$, $E' = \{\{w, u_1\}, \{w, u_{k-1}\} \mid w \in W\} \subseteq \overline{E}$ and consider the subgraph $G' = (V', E')$ of $\overline{G}$. Since every vertex in W is adjacent to u_1 and u_{k-1}, G' is connected. The degree of every vertex in W is two, and the degree of u_1 and u_{k-1} are both $|W|$. Hence, the number of odd-degree vertices in G' is zero or two. Therefore, G' is Eulerian or semi-Eulerian. Since G' has $2|W|$ edges, G' contains a trail T with $2|W| + 1 \geq 2\lceil k/2 \rceil + 1 \geq k+1$ vertices. Since G' is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains T.

▹ Case 3-3: $\lfloor k/2 \rfloor < |U| \leq k-3$.

By the induction hypothesis, in G or $\overline{G}$, there exists a trail T with $|W| \geq 3$ vertices such that every vertex in T is an element of W. Let $T = w_1 e'_1 w_2 e'_2 \ldots e'_{|W|-1} w_{|W|}$ with $e'_i = \{w_i, w_{i+1}\}, i \in \{1, 2, \ldots, |W|-1\}$ and W' be the set of vertices used in T.

We further distinguish two cases according to the containment of T in G or $\overline{G}$.

▹ Case 3-3-1: T is included in G.

Assume that there exists a vertex $u_i \in U$ adjacent to two

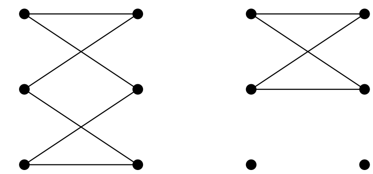

Fig. 5 Two subgraphs of G that can be constructed by $w'_1, w'_2, w'_3, u_x, u_y$ and u_z.

vertices $w_x, w_y \in W'$ where $w_x \neq w_y, x < y$. Then, we have a trail

$$
\begin{aligned}
S' = {} & u_1 e_1 u_2 e_2 \ldots u_i \{u_i, w_x\} w_x \ldots w_y \\
& \{w_y, u_i\} u_i e_i \ldots e_{k-2} u_{k-1}.
\end{aligned}
$$

The number of vertices of S' is at least k. Thus, we only need to consider the case where, for every vertex $u \in U$, there exists at most one element of W' adjacent to u in G.

Let w'_1, w'_2 and w'_3 be any three different vertices of W'. For every vertex $u \in U$, there exists at most one element of W' adjacent to u in G. Then, u is adjacent to at least two vertices of w_1,w_2, and w_3 in $\overline{G}$. Therefore, $\overline{G}$ has the following bipartite graph G' as a subgraph:

- The partite sets of G' are $A = \{w'_1, w'_2, w'_3\}$ and $B = U$;
- The degree of each vertex in B is two.

From Lemma 4, G' has a trail X with $2|B|$ edges, i.e., $2|B| + 1 = 2|U| + 1 > 2 \cdot \lfloor k/2 \rfloor + 1 \geq k$ vertices. Since G' is a subgraph of $\overline{G}$, $\overline{G}$ also has X.

▹ Case 3-3-2: T is included in $\overline{G}$.

Let w'_1, w'_2 and w'_3 be any three different vertices of W'. First, assume that in G there exist three vertices $u_x, u_y, u_z \in U - \{u_1, u_{k-1}\}$ that are adjacent to at least two of the vertices w'_1, w'_2 and w'_3. Then, one of the graphs in Fig. 5 always appears as a subgraph of G. In both cases, there exists a cycle C that has a vertex in U. Therefore, we have a trail $S' = u_1 e_1 u_2 e_2 \ldots e_{x-1} C e_x \ldots e_{k-2} u_{k-1}$, and the number of vertices of S' is at least k.

Second, assume there are at most two elements of $U - \{u_1, u_{k-1}\}$ that are adjacent to at least two vertices of w'_1, w'_2 and w'_3 in G. Let c and d be those two vertices of $U - \{u_1, u_{k-1}\}$. Then, there is a subbipartite graph G' in $\overline{G}$:

- The partite sets of G' are $A = \{w'_1, w'_2, w'_3\}$ and $B = U - \{u_1, u_{k-1}, c, d\}$;
- The degree of each vertex in B is two.

From Lemma 4, G' has a trail X with $2|B|+1 = 2(|U|-4)+1$ vertices. Let $s, t \in W'$ be the endpoints of X. We now construct a trail T' with $|W|$ vertices such that it only consists of the vertices and edges used in T, and does not start at t. If $w_1 \neq t$, then we have $T' = T$. If $w_1 = t$ and T is a circuit, then we have $T' = w_2 e'_2 \ldots e'_{|W|-1} w_{|W|} e'_1 w_2$ since $w_1 = w_{|W|}$ and $w_1 \neq w_2$. If $w_1 = t$ and T is not a circuit, then we have $T' = w_{|W|} e'_{|W|-1} w_{|W|-1} e'_{|W|-2} \ldots e'_1 w_1$ since $w_1 \neq w_{|W|}$. Therefore, T' can be constructed.

Since T is included in $\overline{G}$, T' is also included in $\overline{G}$. Let w, x be the endpoints of T'. Then, we have a trail $Y = X\{t, u_1\}u_1\{u_1, w\}T'\{x, u_{k-1}\}u_{k-1}$ with $2|U| + |W| - 5$ edges. Hence, Y is a trail with $2|U| + |W| - 4 = 2|U| + (k - |U|) - 4 = k + |U| - 4 > k + \lfloor k/2 \rfloor - 4 \geq k$ vertices. □

5. Conclusion

From Theorems 1 and 2, we conclude that $R(\mathcal{T}_k, \mathcal{T}_k) = k$ when $k \leq 6$ and $2\sqrt{k} + \Theta(1) \leq R(\mathcal{T}_k, \mathcal{T}_k) \leq k$ when $k \geq 7$.

Future work is to find stricter upper and lower bounds. Another challenge is to find upper and lower bounds of $R(\mathcal{T}_k, \mathcal{T}_\ell)$ for any k and ℓ.

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